Question 1. Suppose $f(x, y)$ is a non-smooth bounded convex function defined on $\mathbb{R}^1 \times [0, 1]$. Prove that f is independent of x.

Question 2.

Let $f : [0, 1] \to [0, 1]$ be the tent map given by

$$f(x) = \begin{cases} 2x, & \text{when } 0 \le x \le 1/2, \quad \textit{and} \\ 2(1-x), & \text{when } 1/2 < x \le 1. \end{cases}$$

Prove that there is a point $x_0 \in (0, 1)$ such that the orbit of x_0 under f is dense in $[0, 1]$, i.e., the set $\{f^n(x_0)|n = 1, 2, 3, ...\}$ is a dense subset of $[0, 1]$.

Question 3.

Consider the Cauchy problem for the Burger's equation (B):

$$\begin{cases} \partial_t u + u\partial_x u = \epsilon \partial_x^2 u, x \in \mathbf{R}, t > 0 \\ u(x, t = 0) = u_0(x) \end{cases}$$

where $\epsilon \in (0, 1)$ is a constant, and $u_0(x)$ is a smooth periodic function of period 1.

(1) Prove that if $u^\epsilon(x, t)$ is a solution to (B), then $u^\epsilon(x, t)$ is uniformly bounded independent of ϵ.

(2) Prove that the solution $u^\epsilon(x, t)$ is periodic in x with period 1.

(3) Show that if $u^\epsilon(x, t)$ is a solution to (B), then $\partial_x u^\epsilon(x, t) \le \frac{1}{t}$ for all $x \in \mathbf{R}, t > 0$.

(4) Show that total variation of $u^\epsilon(x, t)$ in x is uniformly bounded independent of ϵ for $t > 0$, i.e.

$$TV_{[0,1]} u^\epsilon(x, t) \le C(t).$$

(5)(*) Discuss the convergence property of $u^\epsilon(x, t)$ as $\epsilon \to 0^+$.

Question 4.

Let $D^+ = \{(x, y) : x^2 + y^2 < 1, y > 0\}$. Find all the solutions:

$$\begin{cases} \Delta u &= 0 \text{ in } D^+ \\ u &> 0 \text{ in } D^+ \\ u &= 0 \text{ on } \{(x, y) : x^2 + y^2 = 1, y > 0\} \cup \{(x, 0) : |x| < 1\}. \end{cases}$$